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NON-HOPFIAN GROUPS WITH FULLY INVARIANT KERNELS. I

BY

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ABSTRACT. Let $\mathfrak{L}$ consist of the groups $G(l, m) = (a, b; a^{-1}b^l a = b^m)$ where $|l| \neq 1 \neq |m|$, $lm \neq 0$ and l, m are coprime. We characterize the endomorphisms of these groups, compute the centralizers of special elements and show that the endomorphism $a \to a$, $b \to b^l$ is onto with a nontrivial fully invariant kernel. Hence $G(l, m)$ is non-Hopfian in the 'fully invariant sense.'

Our purpose is to prove the results announced in [1], concerning the endomorphisms of the non-Hopfian one-relator groups $\mathfrak{L}$ found in [2] and isolated by G. Baumslag in [3]. $\mathfrak{L}$ consists of the groups $G(l, m)$ presented by $(a, b; a^{-1}b^l a = b^m)$ where $|l| \neq 1 \neq |m|$, $lm \neq 0$ and l, m are coprime. Let G' denote the normal closure of b in $G(l, m)$ and N the kernel of the endomorphism η: $a \to a$, $b \to b^l$. We will prove

Theorem 1. *If* $\tau: a \to A$, $b \to B \neq 1$ *defines an endomorphism of* $G(l, m)$ *then*
(1) B *is in* G' *and can be written in the form* $B = D^{-1}b^k D$ (D *in* $G(l, m)$).
(2) $DAD^{-1} = ca$ *where* c *is in the centralizer of* b^{lk} *in* G'.

Theorem 2. N *is a proper fully invariant subgroup of* $G(l, m)$ *such that* $G(l, m)/N$ *is isomorphic to* $G(l, m)$.

A general reference for the proofs of these theorems is [4].

1. Basic lemmas.

Lemma 1. η: $a \to a$, $b \to b^l$ *defines an onto endomorphism of* $G(l, m)$ *with nontrivial kernel* N *where* N *is the normal closure of the subgroup generated by*

$$W(a, b) = ([b, a]^t b^s) b^{-1}$$

and

$$V(a, b) = a^{-1} ba ([b, a]^t b^s)^{-m}$$

such that $(m - l)t + ls = 1$.

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Proof. That η is onto and $N \neq 1$ follow from [2]. Let H be presented by $(a, b; a^{-1}b^l a = b^m, W(a, b) = V(a, b) = 1)$. Define $\alpha: G \to H$ where $\alpha: a \to a, b \to b$ and $\beta: H \to G$ where $a \to a$, $b \to b^l$. Now observe that β is an isomorphism (with inverse $\bar{\beta}: H \to G$, $\bar{\beta}: a \to a$, $b \to [b, a]^t b^s$) and that $\alpha\beta = \eta$. Hence $\ker \eta = \ker \alpha\beta = N$.

The proofs of Lemmas 2 and 3 are immediate consequences of the Reidemeister-Schreier rewriting process [4].

Lemma 2. *Let G' be the normal closure of b in $G(l, m)$. $G' = (\cdots x_i \cdots; x_i^l = x_{i+1}^m$, where i runs through the integers) and $x_i = a^i b a^{-i}$ in $G(l, m)$.*

Lemma 3. *In G' we have for $k > 0$,*

$$(x_j)^{lk} = (x_{j+k})^{mk} \text{ for any } j, \tag{1}$$

$$(x_i)^{mk} = (x_{i-k})^{lk} \text{ for any } i. \tag{2}$$

Let G'' denote the commutator subgroup of G' and $Q(l, m)$, the additive subgroup of the rationals, generated by $(l/m)^i$ where i runs through the integers. G/G'' is isomorphic to the obvious split extension of $Q(l, m)$ by an infinite cyclic group. Explicitly

Lemma 4. *G'/G'' is isomorphic to $Q(l, m)$ under the map $y: x_i \to (l/m)^i$.*

2. **Conjugacy in $G(l, m)$.** Here we examine certain subgroups of G' and their length functions to establish certain conjugacy properties.

For $j < k$ let

$$A(j, k) = (x_j, \cdots, x_k; x_j^l = x_{j+1}^m, \cdots, x_{k-1}^l = x_k^m),$$

$$A(j, j) = (x_j; \ ) \text{ the infinite cyclic group,}$$

$$A(-\infty, k) = (\cdots x_{k-1}, x_k; x_i^l = x_{i+1}^m, i < k),$$

$$A(j, +\infty) = (x_j, x_{j+1}, \cdots,; x_i^l = x_{i+1}^m, j \leq i).$$

The *left factorization* of $A(j, k)$ is given by the free product with amalgamation

$$A(j, k-1) * A(k, k), \qquad x_{k-1}^l = x_k^m,$$

and the *right factorization* of $A(j, k)$ by

$$A(j, j) * A(j+1, k), \qquad x_j^l = x_{j+1}^m.$$

For x in $A(j, k)$ let $\lambda_{jk}(x)$ denote the representative length w.r.t. the left factorization and $\rho_{jk}(x)$ the representative length w.r.t. the right factorization

[4]. We will refer to these functions respectively, as the left and right length functions of $A(j, k)$. Elements cyclically reduced w.r.t. these length functions will be referred to as left (resp. right) cyclically reduced. For $p \leq j$, $A(p, k)$ will be called a left extension of $A(j, k)$ and for $q \geq k$, $A(j, q)$ will be called a right extension of $A(j, k)$. For x in G' and $n \geq 0$, $a^{-n}xa^{n}$ (resp. $a^{n}xa^{-n}$) is called a left (resp. right) translate.

In proving results where a dual statement arises by substituting 'right' for 'left' we omit when obvious the proof of the dual. We will refer to Theorem 4.6 in [4] as Theorem 4.6.

The proofs of Corollaries 1 and 2 are immediate consequences of the properties of representative length functions [4].

Corollary 1. *If x is left (resp. right) cyclically reduced in $A(j, k)$ then x is left (resp. right) cyclically reduced in all left (resp. right) extensions of $A(j, k)$ and $\lambda_{jk}(x) = \lambda_{pk}(x)$ (resp. $\rho_{jk}(x) = \rho_{jq}(x)$).*

Corollary 2. *If x is left (resp. right) cyclically reduced in $A(j, k)$ and $\lambda_{jk}(x) \geq 2$ (resp. $\rho_{jk}(x) \geq 2$). Then for $|n| \geq 1$, x^{n} is left (resp. right) cyclically reduced in $A(j, k)$ and $\lambda_{jk}(x^{n}) = |n|\lambda_{jk}(x)$ (resp. $\rho_{jk}(x^{n}) = |n|\rho_{jk}(x)$).*

Corollary 3. *If x is left (resp. right) cyclically reduced in $A(j, k)$ and $\lambda_{jk}(x) \geq 2$ (resp. $\rho_{jk}(x) \geq 2$) then for all left (resp. right) translates y of x^{l} in a given left (resp. right) éxtension of $A(j, k)$, y is not conjugate to x^{m} in that extention.*

Proof. For $y = x^{l}$, by Corollaries 1 and 2, $\lambda_{pk}(x^{l}) \neq \lambda_{pk}(x^{m})$ so, from Theorem 4.6, y and x^{m} are not conjugate in any left extension. For $y = a^{-n}x^{l}a^{n}$, $\lambda_{pk}(y) \leq 1$ and again by Theorem 4.6 the result follows.

An immediate consequence of Theorem 4.6 is

Lemma 5. *Let $G = A *_{H} B$, $H \subset \text{Center}(B)$. If g is cyclically reduced in G and g is conjugate to h in H then g is conjugate in A to h.*

Corollary 4. *If x is left (resp. right) cyclically reduced and $\lambda_{jk}(x) \geq 2$ (resp. $\rho_{jk}(x) \geq 2$) then x is not conjugate in any right (left) extension $A(j, q)$ (resp. $A(p, q)$) to an element in $A(q, q)$ (resp. $A(p, p)$).*

Proof. For $q = k$, we have from Theorem 4.6 and Corollary 2 that x^{n}, $|n| \geq 1$, is not conjugate in $A(j, k)$ to an element in $A(k, k)$. Suppose the result for $q \leq k + i$ and let $q = k + i + 1$ and note $\lambda_{jq}(x) \leq 1$. If x is conjugate in $A(j, q)$ to an element y in $A(q, q)$, y not conjugate in $A(j, q)$ to an element of the amalgamated subgroup H generated by x_{q}^{m} then, by Theorem 4.6, x, y are in $A(q, q)$ and conjugate there. Thus $x = x_{q}^{\mu}$, $\mu \neq 0$. Now by Lemma 3,

$$x_q^{m^{i+1}} = x_{q-(i+1)}^{l^{i+1}} = x_k^{l^{i+1}}$$

so

$$x^{m^{i+1}} = (x_k^{l^{i+1}})^{\mu}$$

which is contrary to x^n not in $A(k, k)$ for $|n| \geq 1$. Thus if x is conjugate in $A(j, q)$ to an element y in $A(q, q)$ then y is conjugate in $A(j, q)$ to an element h in H. Since x is cyclically reduced in $A(j, q)$ and x is in $A(j, k+i)$, by Lemma 5, x would be conjugate to h in $A(j, k+i)$ contrary to our inductive hypothesis.

Corollary 5. *If x and y are left (resp. right) cyclically reduced in $A(j, k)$, $\lambda_{jk}(x) \geq 2$ (resp. $\rho_{jk}(x) \geq 2$) then x is conjugate to y in a right (resp. left) extension of $A(j, k)$ implies x conjugate to y in $A(j, k)$.*

Proof. For $q = k$ the result holds. Suppose the result holds for $q \leq k + i$ and let $q = k + i + 1$. By Corollary 4, x is not conjugate in $A(j, q)$ to an element in $A(q, q)$. Hence, by Theorem 4.6, x is conjugate to y in $A(j, k+i)$. Applying the inductive hypothesis yields x conjugate to y in $A(j, k)$.

Corollary 6. *If x is not conjugate in G' to an element in $A(i, i)$ for any i and x is in $A(j, k)$ then $j < k$ and (1) there exists a j' and an element y such that $j \leq j' < k$, y is cyclically reduced in $A(j', k)$, $\rho_{j'k}(y) \geq 2$ and x is conjugate to y in $A(j, k)$; (2) there exists a k' and an element z such that $j < k' \leq k$, z is cyclically reduced in $A(j, k')$, $\lambda_{jk'}(z) \geq 2$, x is conjugate to z in $A(j, k)$.*

Proof. Clearly $j < k$. For $k - j = 1$ the result holds since x is not conjugate in G' to an element of either $A(j, j)$ or $A(j+1, j+1)$. Thus, cyclically reducing x in $A(j, j+1)$ yields y and setting $y = z$ yields our construction. Suppose the result is true for $k - j \leq n$ and let $k - j = n + 1$. Now cyclically reduce x w.r.t. the right factorization of $A(j, k)$ to y'. If $\rho_{jk}(y') \geq 2$ let $y = y'$. Otherwise y' is in the factor $A(j+1, k)$ and we can apply our inductive hypothesis, obtaining from y' an element y with the desired property. We obtain z similarly.

Corollary 7. *If x is not conjugate in G' to an element in $A(i, i)$ for any i then x^l is not conjugate in $G(l, m)$ to x^m.*

Proof. Let x be in $A(j, k)$ and y, z have the relation to x described in Corollary 6. If x^l is conjugate in $G(l, m)$ to x^m then there is an integer q and an element D in G' such that $a^{-q}x^l a^q = D^{-1}x^m D$. Now there are u, v in $A(j, k)$ such that $x = u^{-1}yu$, $x = v^{-1}zv$. Hence

$$a^{-q}u^{-1}y^{l}ua^{q} = D^{-1}u^{-1}y^{m}uD, \qquad a^{-q}v^{-1}z^{l}va^{q} = D^{-1}v^{-1}z^{m}vD.$$

Transposing and conjugating by a^q yields

$$a^{-q}y^{l}a^{q} = (a^{-q}ua^{q}D^{-1}u^{-1})\,y^{m}\,(uDa^{-q}u^{-1}a^{q}),$$

$$a^{-q}z^{l}a^{q} = (a^{-q}va^{q}D^{-1}v^{-1})\,z^{m}\,(vDa^{-q}v^{-1}a^{q}).$$

Now for some choice of integers e, f and g, h, $uDa^{-q}u^{-1}a^{q}$ is in $A(e, f)$, $vDa^{-q}v^{-1}a^{q}$ is in $A(g, h)$.

Case 1. $q \geq 0$; $a^{-q}z^{l}a^{q}$ is in $A(j-q, k')$. Since z is left cyclically reduced in $A(j, k')$ and $\lambda_{j_k,k'}(z) \geq 2$ by Corollary 3 $a^{-q}z^{l}a^{q}$ is not conjugate in any left extension of $A(j-q, k')$ to z^m. In particular choose $p = \min(g, j-q)$ and also let $r = \max(h, k')$. If x^l is conjugate in $G(l, m)$ to x^m then $a^{-q}z^{l}a^{q}$ is conjugate in $A(p, r)$ to z^m. Since z^m and $a^{-q}z^{l}a^{q}$ are left cyclically reduced in $A(p, k')$ and $\lambda_{p_k,k'}(z^m) \geq 2$, if z^m is conjugate in $A(p, r)$ to $a^{-q}z^{l}a^{q}$ then by Corollary 5, z^m is conjugate in $A(p, k')$ to $a^{-q}z^{l}a^{q}$. However, we have shown that $a^{-q}z^{l}a^{q}$ is not conjugate to z^m in $A(p, k')$ for $p \leq j-q$.

Case 2. $q < 0$; apply an analogous argument on y.

It is immediate from Corollary 7 that

Corollary 8. *If B is in G' then B^l is conjugate to B^m iff B is conjugate to x^k for some i and k.*

3. **The endomorphisms of** $G(l, m)$.

Proof of Theorem 1. Note that the defining relators have a-exponent sum zero so all relators have a-exponent sum zero which puts B in G'. In fact G' and hence G'' are fully invariant. Now $A^{-1}B^{l}A = B^{m}$, so by Corollary 8, B is conjugate in G' to x_c^k and hence in $G(l, m)$ to b^k. Dividing by G'' preserves the a-exponent on A. Every element in $G(l, m)/G''$ has the form ra^n where r is in $Q(l, m)$ of Lemma 4. Conjugation by a^n is seen to act on $Q(l, m)$ as multiplication by $(m/l)^n$. Letting $A \equiv ra^n \bmod G''$ and noting $b^k \equiv k \bmod G''$ reveals that when $A^{-1}B^{l}A = B^{m}$ is viewed $\bmod G''$ the consequence is $a^{-n}rlkra^{n} = mk$ and hence $(m/l)^n k = mk$. Since $B \neq 1$, $k \neq 0$, so we have $n = 1$. Thus $DAD^{-1} = ca$ where c is in G'. Now $a^{-1}c^{-1}b^{lk}ca = b^{mk} = a^{-1}b^{lk}a$ so c is in the centralizer of b^{lk}.

4. **Fully invariant kernels.** The proofs of Lemmas 6 and 7 below follow from Lemma 3 by a straightforward computation.

Lemma 6. *For $n \neq 0$, $k \geq 0$,*

(1) *x_0^n is in $A(k, k)$ iff m^k/n,*

(2) *x_0^n is in $A(-k, -k)$ iff m^k/n.*

Lemma 7. *For each $n \neq 0$ there exist unique integers $p \leq 0 \leq k$ such that x_0^n is in $A(i, i)$ iff $p \leq i \leq k$.*

Lemma 8. *Let $G = A *_H B$. Suppose a is in* Center(A) *and not in H. If $g^{-1}ag$ is in A, then g is in A.*

Proof. If g has representative length zero g is in A. Suppose the result for g of length $< n$ and let $g = hg_1 \cdots g_n$. Note that g_1 is in A for otherwise $g^{-1}ag$ has length $1 + 2n$. Now a in Center(A) and the inductive hypothesis yield the result

Lemma 9. *Let $G = C *_K (A *_H B) = (C *_K A) *_H B$. If a is in* Center(A) *and not in K, H then $g^{-1}ag = a$ implies g in A.*

Proof. View G with respect to the factors C and $A *_H B$. If g has length zero g is in A. Suppose $g = kg_1 \cdots g_n$ and assume the result for lengths $< n$. Note g_1 is in $A *_H B$ since $g^{-1}ag$ has length 1. If $n > 1$ then g_1 is in K for otherwise $g^{-1}ag$ has length ≥ 3 and so by Lemma 8 and the inductive hypothesis the result holds. If $n = 1$, g is in $A *_H B$ and Lemma 8 yields the result.

Lemma 10. *Let H_i be the subgroup generated by x_i^l. The* Center$(A(p, k))$, *$p \leq 0 \leq k$, is $\bigcap_{i=p}^{k-1} Hp$ if $p \neq k$ and $A(p, p)$ for $p = k$.*

Proof. Take induction on $k - p$ applying [4, Corollary 4.5].

Lemma 11. *The centralizer of $(x_0^l)^j$, $j \neq 0$, in G' is $A(p, k)$ where p, k are given in Lemma 7 for $n = lj$.*

Proof. View G' as

$$A(-\infty, p^{-1}) \underset{H_{p-1}}{*} (A(p, k) \underset{H_k}{*} A(k + 1, +\infty))$$

and note x_0^n is not in H_k, H_{p-1}. Hence if x_0^n is in Center$(A(p, k))$ then $A(p, k)$ is the centralizer of x_0^n in G', by Lemma 9. Now x_0^n is in $A(i, i)$ for $p \leq i \leq k$ and so in H_i for $p \leq i \leq k - 1$. By Lemma 10, x_0^n is in Center$(A(p, k))$.

Lemma 12. *Let η be as in Lemma 1. If $n = lj$, $j \neq 0$, and c in G' commutes with b^n then $c\eta$ commutes with b^n.*

Proof. Let $A(p, k)$ be the centralizer of b^n in G' as in Lemma 11 and observe that $A(p, k)\eta \subseteq A(p, k)$.

Proof of Theorem 2. We show for any endomorphism τ, $N\tau \subseteq N$. For τ: $a \rightarrow a$, $b \rightarrow 1$, we have $N\tau = 1$. Suppose τ is as in Theorem 1. Note that $N\tau \subseteq N$ iff $DW(a, b)\tau D^{-1}$ and $DV(a, b)\tau D^{-1}$ are in N. Now

$$DW(a, b)\tau D^{-1} = W(ca, b^k), \qquad DV(a, b)\tau D^{-1} = V(ca, b^k).$$

It follows directly from Lemma 12 that $W(ca, b^k)\eta = 1 = V(ca, b^k)\eta$.

5. **The generalized Hopfian problem.** Let P be a property. G is said to be non-Hopfian in the P-sense iff there is a proper normal subgroup N possessing property P such that G/N is isomorphic to G. Thus the groups in $\mathfrak{L}$ are non-Hopfian in the fully invariant sense. It is pointed out in [1] that reduced free groups are Hopfian in the fully invariant sense. In fact it follows from [4, Theorem 3.3] that free groups of arbitrary rank are Hopfian in the characteristic sense.

Problem. Are there groups Hopfian in the fully invariant sense but non-Hopfian in the characteristic sense ?

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